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CONSTRUCTING THE DISCOUNT FUNCTION WITH THE PARSIMONIOUS EXTENDED NELSON-SIEGEL FUNCTION: EVIDENCE FROM THE NIGERIAN EUROBOND

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Abstract

Nigeria currently carrying out economic reforms is under the surveillance of institutional investors. To construct the market discount function through its yield curve, we applied the extended functional Nelson-Siegel parsimonious function with time-decay parameters to the Eurobond data to employ flexibility of the model and estimate the time varying parameters. In order to present a deep investigation of the Euro-bond market by reason of time to maturity, our contribution is anchored on following objectives: (i) construct the discount function (ii) investigate the limiting behavior of the yield curves and (iii) predict the in-sample yield. The data presented involves the daily closing of the Nigerian Eurobond yield covering January to December 2021 and 2022. The data was fitted to the observed Nigerian Eurobond yield curve to model the discount function. The ordinary least square method is used for the analysis and the estimated parameters were used to compute the in-sample yield. Test of goodness of fit was conducted showing that the model fits in well to the observed data demonstrated by the model's R-square adjusted through the predicted yields after obtaining the two decay factors. This paper has implications on life insurance products associated with minimum guaranteed benefit schemes for the insured. Based on the terms and conditions of the contract, the insured pays regular premium invested in debt instruments and receives benefit at death or at maturity of the policy depending on the market performance of the fund. There is a guaranteed benefit which the insured earns irrespective of the performance of the life fund. The insurer would then pay the guaranteed amount even if the benefit eventually drops below the guaranteed amount. Given the discount $\delta(\tau)$ and benefit $\emptyset$, the present value of future death benefits is modeled as $PVFB = \emptyset_1\delta(1) + \emptyset_2\delta(2) + \cdots + \emptyset_{T-1}\delta(T-1) + \emptyset_T\delta(T)$.

Keywords: Consol, Extended Nelson-Siegel, Eurobond, Parsimonious, Yield Curve

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1. INTRODUCTION

This study is concerned with the estimation of the nominal yield curve and the associated discount function by means of integral transform approach while focusing on a model of the forward rate curve from where the functional form of the yield curve could be derived by integrating the specified forward rate function. This technique could be explained as a generalization of the commonly employed parsimonious form of the yield curve estimation initially proposed in Nelson and Siegel (1987). Valuable market information is entrenched in the yield curve such as market's expectation on the future trajectory of interest rate analysis. Yield curves are not directly observable despite the availability of prices and the yields of bonds in the market. To obtain a yield curve directly from the market data, a zero coupon bond over a continuum of maturities will be required. Although bonds are issued over a defined set of maturities, many bonds for longer term maturities are fixed coupon bonds. Consequently, effort to extract yield curve from the bond market requires modelling zero rates.

The yield curve considered here will be specifically based on the extended Nelson-Siegel model which can be suited to the less liquid and less developed markets similar to the Nigerian market. The extended Nelson-Siegel model represents a zero coupon parsimonious model which accounts for distinct deformations of the yield curves to enable a dynamic evaluation of the market with time varying parameters which can be estimated from the market data and representing the curve as a smooth surface.

It has been assumed that a similar interest rate is applied for discounting cash inflows or outflows for all maturities. However, this assumption does not always hold in practice since the rates applied for discounting a series of cash-flows at various maturities vary. This can be verified by contrasting the actual interest rates on zero coupon bonds which disburses a single payment at maturity under no intermediate coupon payments. Consequently, these bonds are usually discounted at varying rates based on their remaining term to maturity. The term structure of interest rates represents the connection between yields on comparable financial instruments of varying degree of maturities. A spot rate S_k for maturity k periods is an interest rate payable on a debt instrument of maturity k periods which starts immediately and accumulates interest to maturity, $k \in Z^+$. However, a single forward rate $f(\xi)$ is an interest rate payable on a future debt which begins at time ξ to $\xi+1$; $\xi = 0,1,2,3,...$ while a short interest rate is a rate applied to a short interval of time up to a year including instantaneous rate over an infinitesimal interval of time. Suppose we apply one year as short interest rate to derive forward rate function, we obtain

$$(1+S_k)^k = (1+f(1))(1+f(2))(1+f(3))\times...\times(1+f(k-1)) \tag{1}$$

or equivalently;

$$\left(1+f\left(k-1\right)\right)=\frac{\left(1+S_k\right)^k}{\left(1+S_{k-1}\right)^{k-1}} \tag{2}$$

From the continuously compounded interest rate perspective, we let $\delta\left(\xi\right)$ be the spot force of interest and $\Phi\left(\xi\right)$ be the forward force of interest to obtain a functional form as

$$e^{\xi\times\delta(\xi)}=e^{\int_0^\xi \Phi(u)du} \tag{3}$$

$$\xi\delta\left(\xi\right)=\int_0^\xi \Phi\left(u\right)du \Rightarrow \delta\left(\xi\right)=\frac{1}{\xi}\int_0^\xi \Phi\left(u\right)du \tag{4}$$

Differentiating both sides we have the first order differential equation

$$\Phi\left(\xi\right)=\xi\frac{d}{d\xi}\delta\left(\xi\right)+\delta\left(\xi\right) \tag{5}$$

2. LITERATURE REVIEW

In Castello and Resta (2019), the trajectory motion of term structure is defined by the yield curves which are potentially obtained from a large number of prices of debt instruments. From the author's view, we see that their information content could be encapsulated into a parsimonious number of parameters provided the particular trend shown by these curves is identified. Hence the yield curve describes the correspondence between interest rates payable on debt instruments with varying time to maturity spectrum. Nigeria is currently under the surveillance of institutional investors and the recent reforms carried out in the financial sector as part of the globalization process constitute an avalanche of economic uncertainties. Consequently, examining its yield curve is markedly significant to the regulators, government, financial institutions and institutional investors to design pricing and hedging strategies against those financial risks so as to quantify financial risks associated with portfolios. The yield curve provides the requisite information to elucidate the fiscal policies that could be explained in terms of the market expectation of fiscal policies and financial market operations over short, medium and long time spectrum so as to design economic strategies.

As observed in Javier (2009), the yield curve therefore elicits instrumental economic information in terms of fiscal inputs to simplify institutional regulation such that when the nominal yield is estimated, the term structure of interest rate will be numerically modelled. The term structure of interest rates is usually obtained from the optimized instantaneous forward rates through technical numerical integration techniques although irrespective of the numerical process applied, a major limitation associated with modelling yield curves seems to anchor on the argument that they usually reflect as many relevant movements in the underlying term structure of interest rates. The rationale behind the functional technique is to derive continuous forward rate function analytically such that the constructed analytical discount bond prices will fit the observed smooth bond price function.

The models developed in Nelson-Siegel (1987) class, in particular the extended Nelson-Siegel in Svensson (1994) assume a continuous function for the instantaneous forward rates which seem to satisfy the solution of a second order differential equation with constant coefficients. The yield curve as observed in Javier (2009) could elicit further economic information in the form of fiscal inputs to ease institutional regulation such that the moment the nominal yield is computed, the term structure of interest rate will be numerically modelled.

In Saunders and Cornett (2014); Castello and Resta (2019); Stelios and Avdoulas (2020), the term structure of interest rate measures the investor's return in a market income instrument. Although extensive works in Chakroun and Abid (2014); Filipovic (2009); Diebold and Rudenbusch (2013); Walli and Bari (2018) Hess (2020) ranging from deterministic to stochastic techniques have been researched on the term structure of interest rates, their discussions have been impressively centered on the advanced economies in America, Canada and Europe but regrettably these authors have not deeply examined the emerging economies like Nigerian markets. The apparent dearth of contributions on the emerging economies in view of their potential fiscal strength capabilities seems to be myopically obscured by these researchers because the trajectories of the yield curve of these economies appear to exhibit some levels of volatilities with frequent market humps as opposed to the yield curve of advanced economies where the yield function is less volatile. Nigeria is embarking on some key economic reforms and the current unabated cash crunch are drivers of financial risks. Furthermore, the apparent dearth of rich academic literature on the term structure plaguing few emerging economies as argued in Chakroun and Abid (2014); Walli and Bari (2018); Stuart (2020) marks the evolution of this study. Although zero coupon yield rates are commercially furnished by market data vendors, they are usually nontransparent and hence may not be trustworthy.

3. METHODOLOGY

As a big market player in Africa, it is observed that Nigeria does not have any standard technique of measuring its term structure up till now through yield curves. As a result of this gap, we shed light on the extended Nelson-Siegel parsimonious function that is both smooth and flexible to construct a computational pool of yield and forward curve trajectories from where the discount and spot rate functions could be derived.

The first approach is to construct an analytical form of the yield curve and then derive the particular yield curve corresponding to the extended Nelson-Siegel. This yield curve function is then used to derive the console and the short rates.

Suppose $f(\tau,\beta)$ defines the forward rate trajectories at current time τ with maturity β. We can partition the interval (τ,β) into k equal sub-intervals such that

$$k \times \Delta\tau = \beta - \tau \tag{6}$$

$$f(\tau,\beta) = \lim_{k\to\infty} \left\{ \begin{array}{l} f(\tau,\tau+\Delta\tau)\dfrac{\Delta\tau}{\beta-\tau} + f(\tau+\Delta\tau,\tau+2\Delta\tau)\dfrac{\Delta\tau}{\beta-\tau} \\ + f(\tau+2\Delta\tau,\tau+3\Delta\tau)\dfrac{\Delta\tau}{\beta-\tau} \\ + f(\tau+(k-1)\Delta\tau,\tau+k\Delta\tau)\dfrac{\Delta\tau}{\beta-\tau} \end{array} \right\} \tag{7}$$

$$f(\tau,\beta) = \frac{1}{\beta-\tau}\sum_{j=0}^{k-1} f\big(\tau+j\Delta\tau,\tau(j+1)\Delta\tau\big)\Delta\tau \tag{8}$$

$$f(\tau,\beta) = \lim_{k\to\infty}\left\{\frac{1}{\beta-\tau}\sum_{j=0}^{k-1} f\big(\tau+j\Delta\tau,\tau(j+1)\Delta\tau\big)\Delta\tau\right\} \tag{9}$$

$$f(\tau,\beta) = \frac{1}{\xi-\tau}\int_{\tau}^{\xi} f(u,u)\,du \tag{10}$$

For the extended Nelson-Siegel function, the forward rate function is defined as

$$f(\tau) = \beta_0 + \beta_1 e^{-\frac{\tau}{\lambda_1}} + \beta_2\left(\frac{\tau}{\lambda_1}e^{-\frac{\tau}{\lambda_1}}\right) + \beta_3\left(\frac{\tau}{\lambda_2}e^{-\frac{\tau}{\lambda_2}}\right) \tag{11}$$

The parameter β_0 defines the asymptote of the zero coupon yield trajectory and represents the long run level of interest rates. The factor β_1 describes the deviation of the forward function from the asymptotic level and represents the variation between long term and short term instantaneous forward rates. The factors β_2 and β_3 govern the magnitudes and directions of humps. The factors λ_1 and λ_2

govern the convergence speed of the exponential term and control the maturity at which the medium term rates approach their maximum. Extreme values of $\{\lambda_1,\lambda_2\}$

would result in slow decay and determine better fit at long maturity but not in the short term when visible curvatures are apparent. Nevertheless low values of $\{\lambda_1, \lambda_2\}$ decay quickly and consequently a better fit for short maturity but not in the long run.

Estimating the parameters of the extended Nelson-Siegel essentially requires transforming the instantaneous forward rate function into zero rates to price coupon bearing bonds. The zero rates is obtained by evaluating the integral of the instantaneous forward rates in equation (11)

Using equation (10), we then construct the yield function

$$y_\tau(\tau) = \frac{1}{\tau}\int_0^\tau \left[\beta_0 + \beta_1 e^{-\frac{u}{\lambda_1}} + \beta_2\left(\frac{u}{\lambda_1}e^{-\frac{u}{\lambda_1}}\right) + \beta_3\left(\frac{u}{\lambda_2}e^{-\frac{u}{\lambda_2}}\right)\right]du \tag{12}$$

$$y_\tau(\tau) =$$
$$\frac{1}{\tau}\times\beta_0\int_0^\tau du + \frac{1}{\tau}\times\beta_1\int_0^\tau e^{-\frac{u}{\lambda_1}}du + \frac{1}{\tau\lambda_1}\times\beta_2\int_0^\tau\left(ue^{-\frac{u}{\lambda_1}}\right)du + \frac{1}{\tau\lambda_2}\times\beta_3\int_0^\tau ue^{-\frac{u}{\lambda_2}}du \tag{13}$$

$$y_\tau(\tau) = \frac{\beta_0}{\tau}\left[u\right]_0^\tau + \frac{\beta_1}{\tau}\left[\frac{e^{-\frac{u}{\lambda 1}}}{\frac{-1}{\lambda_1}}\right]_0^\tau$$
$$+\frac{\beta_2}{\tau\lambda_1}\left(\left[\frac{ue^{-\frac{u}{\lambda_1}}}{-\frac{1}{\lambda_1}}\right]_0^\tau - \int_0^\tau \frac{e^{-\frac{u}{\lambda_1}}}{\frac{-1}{\lambda_1}}du\right) + \frac{\beta_3}{\tau\lambda_2}\left\{\left[\frac{ue^{-\frac{u}{\lambda_2}}}{-\frac{1}{\lambda_2}}\right]_0^\tau - \int_0^\tau \frac{e^{-\frac{u}{\lambda_2}}}{\frac{-1}{\lambda_2}}du\right\} \tag{14}$$

$$y_\tau(\tau) = \frac{\beta_0}{\tau}\left[u\right]_0^\tau - \frac{\beta_1}{\tau}\left[\frac{e^{-\frac{u}{\lambda 1}}}{\frac{1}{\lambda_1}}\right]_0^\tau - \frac{\beta_2}{\tau\lambda_1}\times\left[\frac{ue^{-\frac{u}{\lambda_1}}}{\frac{1}{\lambda_1}}\right]_0^\tau + \frac{\beta_2}{\tau\lambda_1}\times\int_0^\tau\frac{e^{-\frac{u}{\lambda_1}}}{\frac{1}{\lambda_1}}du$$
$$-\frac{\beta_3}{\tau\lambda_2}\times\left[\frac{ue^{-\frac{u}{\lambda_2}}}{\frac{1}{\lambda_2}}\right]_0^\tau + \frac{\beta_3}{\tau\lambda_2}\times\int_0^\tau\frac{e^{-\frac{u}{\lambda_2}}}{\frac{1}{\lambda_2}}du \tag{15}$$

$$y_\tau(\tau) = \frac{\beta_0}{\tau}\left[u\right]_0^\tau - \frac{\beta_1}{\tau}\left[\frac{e^{-\frac{u}{\lambda 1}}}{\frac{1}{\lambda_1}}\right]_0^\tau - \frac{\beta_2}{\tau\lambda_1}\times\left[\frac{ue^{-\frac{u}{\lambda_1}}}{\frac{1}{\lambda_1}}\right]_0^\tau + \frac{\beta_2}{\tau}\times\int_0^\tau e^{-\frac{u}{\lambda_1}}du$$
$$-\frac{\beta_3}{\tau\lambda_2}\times\left[\frac{ue^{-\frac{u}{\lambda_2}}}{\frac{1}{\lambda_2}}\right]_0^\tau + \frac{\beta_3}{\tau}\times\int_0^\tau e^{-\frac{u}{\lambda_2}}du \tag{16}$$

$$y_\tau(\tau)=\frac{\beta_0}{\tau}\left[u\right]_0^\tau-\frac{\beta_1}{\tau}\left[\frac{e^{-\frac{u}{\lambda 1}}}{\frac{1}{\lambda_1}}\right]_0^\tau-\frac{\beta_2}{\tau\lambda_1}\times\left[\frac{ue^{-\frac{u}{\lambda_1}}}{\frac{1}{\lambda_1}}\right]_0^\tau+\frac{\beta_2}{\tau}\times\left[\frac{e^{-\frac{u}{\lambda_1}}}{-\frac{1}{\lambda_1}}\right]_0^\tau$$
$$-\frac{\beta_3}{\tau\lambda_2}\times\left[\frac{ue^{-\frac{u}{\lambda_2}}}{\frac{1}{\lambda_2}}\right]_0^\tau+\frac{\beta_3}{\tau}\times\left[\frac{e^{-\frac{u}{\lambda_2}}}{-\frac{1}{\lambda_2}}\right]_0^\tau \tag{17}$$

$$y_\tau(\tau)=\frac{\beta_0}{\tau}\left[\tau-0\right]-\frac{\beta_1}{\tau}\left[\frac{e^{-\frac{\tau}{\lambda 1}}}{\frac{1}{\lambda_1}}-\frac{1}{\frac{1}{\lambda_1}}\right]-\frac{\beta_2}{\tau\lambda_1}\times\left[\frac{\tau e^{-\frac{\tau}{\lambda_1}}}{\frac{1}{\lambda_1}}-0\right]$$
$$-\frac{\beta_2}{\tau}\times\left[\frac{e^{-\frac{\tau}{\lambda_1}}}{\frac{1}{\lambda_1}}-\frac{1}{\frac{1}{\lambda_1}}\right]-\frac{\beta_3}{\tau\lambda_2}\times\left[\frac{\tau e^{-\frac{\tau}{\lambda_2}}}{\frac{1}{\lambda_2}}-0\right]-\frac{\beta_3}{\tau}\times\left[\frac{e^{-\frac{\tau}{\lambda_2}}}{\frac{1}{\lambda_2}}-\frac{1}{\frac{1}{\lambda_2}}\right] \tag{18}$$

$$y_\tau(\tau)=\beta_0-\frac{\beta_1}{\tau}\left[\lambda_1 e^{-\frac{\tau}{\lambda 1}}-\lambda_1\right]-\frac{\beta_2}{\tau\lambda_1}\times\tau\lambda_1 e^{-\frac{\tau}{\lambda_1}}-\frac{\beta_2}{\tau}\times\left[\lambda_1 e^{-\frac{\tau}{\lambda_1}}-\lambda_1\right]$$
$$-\frac{\beta_3}{\tau\lambda_2}\times\lambda_2\tau e^{-\frac{\tau}{\lambda_2}}-\frac{\beta_3}{\tau}\times\left[\lambda_2 e^{-\frac{\tau}{\lambda_2}}-\lambda_2\right] \tag{19}$$

$$y_\tau(\tau)=\beta_0-\frac{\lambda_1\beta_1}{\tau}\left[e^{-\frac{\tau}{\lambda 1}}-1\right]-\beta_2 e^{-\frac{\tau}{\lambda_1}}-\frac{\lambda_1\beta_2}{\tau}\left[e^{-\frac{\tau}{\lambda_1}}-1\right]$$
$$-\beta_3 e^{-\frac{\tau}{\lambda_2}}-\frac{\lambda_2\beta_3}{\tau}\left[e^{-\frac{\tau}{\lambda_2}}-1\right] \tag{20}$$

The equation (20) can then be applied to derive the console and the short rates.

3.1. The Console's Theorem

$$\lim_{\tau\to\infty} y_\tau\left(\tau\right)=\beta_0 \tag{21}$$

Proof

The yield is given by

$$y_\tau(\tau)=\beta_0-\frac{\lambda_1\beta_1}{\tau}\left[e^{-\frac{\tau}{\lambda 1}}-1\right]-\beta_2 e^{-\frac{\tau}{\lambda_1}}-\frac{\lambda_1\beta_2}{\tau}\left[e^{-\frac{\tau}{\lambda_1}}-1\right]$$
$$-\beta_3 e^{-\frac{\tau}{\lambda_2}}-\frac{\lambda_2\beta_3}{\tau}\left[e^{-\frac{\tau}{\lambda_2}}-1\right] \quad (22)$$

We need to take limit at infinity

$$\lim_{\tau\to\infty} y_\tau(\tau)=\lim_{\tau\to\infty}\beta_0-\lim_{\tau\to\infty}\frac{\lambda_1\beta_1}{\tau}\times\lim_{\tau\to\infty}\left[e^{-\frac{\tau}{\lambda 1}}-1\right]-\beta_2\lim_{\tau\to\infty}e^{-\frac{\tau}{\lambda_1}}$$
$$-\lim_{\tau\to\infty}\frac{\lambda_1\beta_2}{\tau}\times\lim_{\tau\to\infty}\left[e^{-\frac{\tau}{\lambda_1}}-1\right]$$
$$-\beta_3\lim_{\tau\to\infty}e^{-\frac{\tau}{\lambda_2}}-\lim_{\tau\to\infty}\frac{\lambda_2\beta_3}{\tau}\times\lim_{\tau\to\infty}\left[e^{-\frac{\tau}{\lambda_2}}-1\right] \quad (23)$$

$$\lim_{\tau\to\infty} y_\tau(\tau)=\beta_0-0\times(-1)-\beta_2\times 0-0\times(-1)-\beta_3\times(0)-0\times(-1) \quad (24)$$

$$\lim_{\tau\to\infty} y_\tau(\tau)=\beta_0 \quad (25)$$

Q.E.D

3.2. The Short Rate Theorem

The short term rate

$$\lim y_\tau(\tau)=\beta_0-\beta_2-\beta_3 \quad (26)$$

Proof

The yield curve rate is

$$y_\tau(\tau)=\beta_0-\frac{\lambda_1\beta_1}{\tau}\left[e^{-\frac{\tau}{\lambda 1}}-1\right]-\beta_2 e^{-\frac{\tau}{\lambda_1}}-\frac{\lambda_1\beta_2}{\tau}\left[e^{-\frac{\tau}{\lambda_1}}-1\right]$$
$$-\beta_3 e^{-\frac{\tau}{\lambda_2}}-\frac{\lambda_2\beta_3}{\tau}\left[e^{-\frac{\tau}{\lambda_2}}-1\right] \quad (27)$$

$$\lim_{\tau\to 0} y_\tau(\tau) = \lim_{\tau\to 0}\beta_0 - \lim_{\tau\to 0}\frac{\lambda_1\beta_1}{\tau}\times\lim_{\tau\to 0}\left[e^{-\frac{\tau}{\lambda 1}}-1\right] - \lim_{\tau\to 0}\beta_2 e^{-\frac{\tau}{\lambda_1}}$$
$$-\lim_{\tau\to 0}\frac{\lambda_1\beta_2}{\tau}\times\lim_{\tau\to 0}\left[e^{-\frac{\tau}{\lambda_1}}-1\right]$$
$$-\lim_{\tau\to 0}\beta_3 e^{-\frac{\tau}{\lambda_2}} - \lim_{\tau\to 0}\frac{\lambda_2\beta_3}{\tau}\times\lim_{\tau\to 0}\left[e^{-\frac{\tau}{\lambda_2}}-1\right] \quad (28)$$

Applying the L'Hopital rule to the quotient terms only and take the limit at zero, we have

$$\lim_{\tau\to 0} y_\tau(\tau) = \lim_{\tau\to 0}\beta_0 - \lim_{\tau\to 0}\frac{0}{1}\times\lim_{\tau\to 0}\left[-\frac{1}{\lambda_1}e^{-\frac{\tau}{\lambda 1}}\right] - \lim_{\tau\to 0}\beta_2 e^{-\frac{\tau}{\lambda_1}}$$
$$-\lim_{\tau\to 0}\frac{0}{1}\times\lim_{\tau\to 0}\left[-\frac{1}{\lambda_1}e^{-\frac{\tau}{\lambda_1}}\right]$$
$$-\lim_{\tau\to 0}\beta_3 e^{-\frac{\tau}{\lambda_2}} - \lim_{\tau\to 0}\frac{0}{1}\times\lim_{\tau\to 0}\left[-\frac{1}{\lambda_2}e^{-\frac{\tau}{\lambda_2}}\right] \quad (29)$$

$$\lim_{\tau\to 0} y_\tau(\tau) = \beta_0 - \beta_2 - \beta_3 \quad (30)$$

Thus the difference between the console and the short rate is given as

$$\lim_{\tau\to\infty} y_\tau(\tau) - \lim_{\tau\to 0} y_\tau(\tau) = \beta_2 + \beta_3 \quad (31)$$

Q.E.D

Following Diebold and Li (2006), we obtain lamda as $\lambda_1 = 0.03778$ and $\lambda_2 = 0.0669$ through unconstrained optimization.

3.3. Application in Life Insurance Death Benefits

Suppose an insured purchases a term insurance with maturity date T. The insured receives death benefits at the end of the year of death when the assured dies between year m and year $m+1$. The death benefit is the higher of the investment at time zero and the fund value at time of death. However, applying the actuarial assumption to the death benefit, then $\int_0^\infty l_{x+t}\mu_{x+t}dt$ lives would die per unit scheme with probability

$F_{T(x)}(t) = 1 - \exp\left(-\int_0^t \mu_{x+\xi} d\xi\right)$. The benefit at time t payable upon death will be the greater value of G_t and S_t

$$G_t = S_0 e^{gt} \tag{32}$$

The benefit ϕ_t at time t is given as

$$\phi_t = \begin{cases} S_t & if \quad S_t > G_t \\ G_t & if \quad S_t < G_t \end{cases} \tag{33}$$

$$\phi_t = \max(G_t, S_t) \tag{34}$$

$$\phi_t = \max(G_t - S_t, 0) + S_t \tag{35}$$

$$\phi_t = (G_t - S_t)^+ + S_t \tag{36}$$

G_t is the death benefit guaranteed level interest rate g per unit investment over time while S_t is the value of the underlying equity investment at time t representing the accumulation function from time 0 to t where we assume $S_0 = 1$.

3.4. The Present Value of Future Benefits

This paper has implications on life assurance products marketed by life offices associated with minimum guaranteed benefits schemes for the assured. Based on the terms and conditions of the contract, the assured life pays regular premium investible in market debt instruments and consequently, receives benefit at death or at maturity of the policy depending on the market performance of the fund. Nevertheless, there is a guaranteed benefit which the assured earns irrespective of the performance of the life fund. The life office is then obliged to pay the guaranteed sum even if the benefit at maturity or death eventually becomes smaller than the guaranteed sum, although this represents a risk that is usually built into the contract. The present value of the future death benefits is expressed as

$$PVFB = \phi_1\delta(1) + \phi_2\delta(2) + ... + \phi_{T-1}\delta(T-1) + \phi_T\delta(T) \tag{37}$$

where $\delta(\tau)$ is the discount rate function and ϕ is the benefit. The discount rate shows the needed rate of return from the bond. The needed rate of return is the addition of the yield on bonds which are free of default uncertainties and a risk premium that characterizes the default risk of the bond being valued.

$$PVFB=\left\{\left(G_1-S_1\right)^{+}+S_1\right\}\delta\left(1\right)+\left\{\left(G_2-S_2\right)^{+}+S_2\right\}\delta\left(2\right)+\ldots$$
$$+\left\{\left(G_{T-1}-S_{T-1}\right)^{+}+S_{T-1}\right\}\delta\left(T-1\right)+\left\{\left(G_T-S_T\right)^{+}+S_T\right\}\delta\left(T\right) \quad (38)$$

The expectation of $PVFB=\sum_{k=1}^{T}\phi_k\delta\left(k\right)$ under the risk neutral measure $\mathbf{Q}$ becomes

$$APVFB=\sum_{k=1}^{T}\mathbf{E}^{\mathbf{Q}}\phi_k\delta\left(k\right)\left({}_{k-1}q_x\right) \quad (39)$$

$$APVFB=\sum_{k=1}^{T}\mathbf{E}^{\mathbf{Q}}\phi_k\delta\left(k\right)\left(\frac{l_x-l_{x+k-1}}{l_x}\right) \quad (40)$$

where

$$l_x=\int_0^{\infty}l_{x+t}\mu_{x+t}dt \quad (41)$$

is the survival function at age x

$$APVFB=\sum_{K=1}^{T}\mathbf{E}^{\mathbf{Q}}\phi_k\delta\left(k\right)-\frac{1}{l_x}\sum_{K=1}^{T}\mathbf{E}^{\mathbf{Q}}\phi_k\delta\left(k\right)l_{x+k-1} \quad (42)$$

$$APVFB$$
$$=\sum_{K=1}^{T}\mathbf{E}^{\mathbf{Q}}\left[\left(G_t-S_t\right)^{+}+S_t\right]\delta\left(k\right)-\frac{1}{l_x}\sum_{k=1}^{T}\mathbf{E}^{\mathbf{Q}}\left[\left(G_t-S_t\right)^{t}+S_t\right]\delta\left(k\right)l_{x+k-1} \quad (43)$$

$$APVFB=\sum_{K=1}^{T}\left\{\begin{array}{c}\mathbf{E}^{\mathbf{Q}}\left[\left(G_t-S_t\right)^{+}+S_t\right]\delta\left(k\right)-\\ \dfrac{1}{\left(\int_0^{\infty}l_{x+t}\mu_{x+t}dt\right)}\sum_{k=1}^{T}\mathbf{E}^{\mathbf{Q}}\left[\left(G_t-S_t\right)^{+}+S_t\right]\delta\left(k\right)\times l_0e^{-\int_0^{x+k-1}\mu_t dt}\end{array}\right\} \quad (44)$$

where $\mu(x) = GM(1,2)$ is the chosen mortality intensity

$$APVFB = \sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^t + S_t\right] e^{-ky(k)} -$$

$$\left\{ \frac{1}{\left(\int_0^{\infty} l_{x+t} \mu_{x+t} dt \right)} \sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^t + S_t\right] e^{-ky(k)} \times e^{-\int_0^{x+k-1} \mu_t dt} \right\} \quad (45)$$

where

$$y_t(\tau) = \beta_0 \tau + \beta_1 \tau \left[\frac{1 - e^{-\frac{\tau}{\lambda_1}}}{\frac{\tau}{\lambda_1}} \right] + \beta_2 \tau \left[\frac{1 - e^{-\frac{\tau}{\lambda}}}{\frac{\tau}{\lambda_1}} - e^{-\frac{\tau}{\lambda_1}} \right]$$

$$+ \beta_3 \tau \left[\frac{1 - e^{-\frac{\tau}{\lambda_2}}}{\frac{\tau}{\lambda_2}} - e^{-\frac{\tau}{\lambda_2}} \right] \quad (46)$$

$$APVFB = \frac{\sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^+ + S_t\right]}{e^{ky_t(k)}}$$

$$- \frac{\sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^+ + S_t\right]}{\left(\int_0^{\infty} l_{x+t} \mu_{x+t} dt \right) e^{ky(k)}} \times e^{-\left(\int_x^{x+k-1} \mu_t dt \right)} \quad (47)$$

Suppose $GM(1,2)$ is the mortality intensity $\mu(x) = A + BC^x$ where A, B and C are parameters of the mortality law, then the integrated hazard of mortality is

$$\int_x^{x+k-1} \mu_t dt = \int_x^{x+k-1} \left(A + BC^t \right) dt \quad (48)$$

$$\int_x^{x+k-1} \mu_t dt = \left[At + \frac{BC^t}{\log_e C} \right]_x^{x+k-1} \quad (49)$$

$$\int_{x}^{x+k-1} \mu_t dt = A(x+k-1) - Ax + \frac{BC^{x+k-1}}{\log_e C} - \frac{BC^{x}}{\log_e C} \quad (50)$$

$$\int_{x}^{x+k-1} \mu_t dt = Ax + Ak - A - Ax + \frac{BC^{x}}{\log_e C}\left(C^{k-1} - 1\right) \quad (51)$$

The total severity is obtained as

$$\int_{x}^{x+k-1} \mu_t dt = A(K-1) + \frac{Bc^{x}}{\log_e C}\left(C^{k-1} - 1\right) \quad (52)$$

$$APVFB = \sum_{K-1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^{+} + S_t\right] - \sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^{+} + S_t\right] e^{-ky(k)} \times e^{-\left\{A(k-1) + \frac{BC^{x}}{\log_e C}\left(\ell^{k-1} - 1\right)\right\}} \quad (53)$$

$$APVFB = \frac{1}{e^{ky_t(k)}} \left\{ \begin{array}{l} \sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^{+} + S_t\right] \\ -\sum_{k=1}^{T} \mathbf{E}^{\mathbf{Q}}\left[\left(G_t - S_t\right)^{+} + S_t\right] \times e^{-\left\{A(K-1) + \frac{Bc^{x}}{\log_e C}\left(C^{k-1} - 1\right)\right\}} \end{array} \right\} \quad (54)$$

3.5. Estimation and Data Analysis

The classical term structure hypothesis suggests that the smooth yield curve $y(\tau)$ be estimated from the observed bond prices. Recently, the common approach is to compute the implicit forward rates needed to price successive longer maturity bonds at the observed maturities. The smoothed forward rate function is hence derived by fitting a parsimonious functional form to the unsmoothed rates. The unsmoothed forward rates could be transformed to the unsmoothed yield through averaging. We observe currently that minimizing the price errors ultimately leads to grossly large yield errors for bonds particularly bonds with short and medium maturities. This is because prices are most insensitive to yields for short maturities.

Consequently, it is more reasonable to estimate parameters so as to minimize yield errors. In financial modelling, the emphasis is on interest rates rather than prices and as such, it is more reasonable to minimize errors in the yield rather than minimizing errors in the price. To obtain the parameters of the extended Nelson-Siegel model, we estimate the parameters as such to minimize the sum of squared errors between the estimated yields y^E and observed yields to maturity y and hence we can write $\hat{\theta}_i = \arg\min_{\theta_i} \sum_{i=1}^{p} \left(y_i^E - y_i \right)^2$.

The data presented involves the daily closing of the Nigerian Eurobond yield which comprises data from January to December 2021 and 2022 to be analyzed using descriptive statistics in fitting the observed Nigerian Eurobond yield curve under the extended Nelson-Siegel model structure. The data for the twelve months for both 2021 and 2022 were analyzed and the resulting findings were also depicted in curves and tabular forms to incorporate other statistics not captured on the curve and also to enhance easy access to the statistical values and corresponding graph are shown in Table 1 and figure 1 for the first three quarters.

Table 1: First three quarters descriptive statistics-2021

Tenors (τ)	N	Minimum	Maximum	Mean	Std. Deviation
One year	166	.84	6.22	2.0292	.64054
Two years	166	2.26	2.98	2.6323	.15806
Four years	166	3.88	4.82	4.2375	.15617
Six years	166	4.71	5.83	5.2816	.20958
Nine years	166	5.67	7.04	6.2206	.20969
Ten years	166	6.14	7.24	6.6915	.21741
Eleven years	166	6.24	7.43	6.7980	.21563
Seventeen years	166	6.80	7.88	7.3502	.20490
Twenty-six years	166	6.95	7.97	7.4861	.20316
Twenty-eight years	166	7.39	8.42	7.8931	.15832

Source: Authors' computation

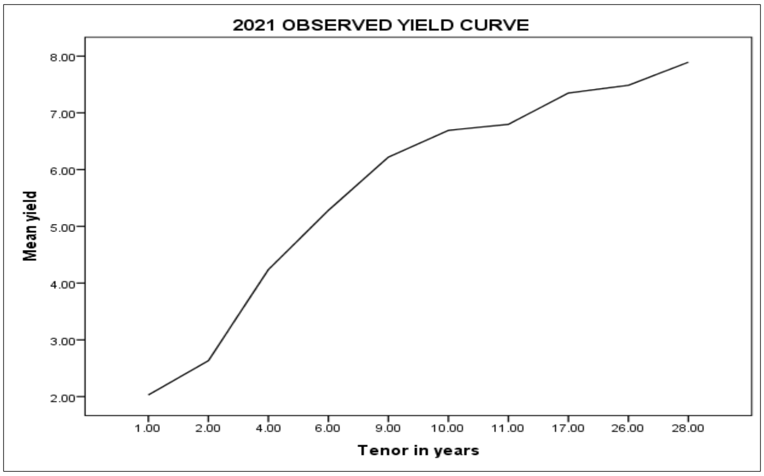

Figure 1: First three quarters' yield curve

This observed data analyzed descriptively consisted of the first three quarters of the year 2021. The data was separated from the fourth quarter due to the increase of additional tenors as recorded. The three quarters contained 10 tenors which are one year, two years, four years, six years, nine years, ten years, eleven years, seventeen years, twenty-six years and twenty-eight with corresponding yield of

$$\left\{\begin{matrix} 2.0292,\ 2.6323,\ 4.2375,\ 5.2816,\ 6.2206,\ 6.6915,\ 6.7980,\ 7.3502, \\ 7.4861\ \textit{and}\ 7.8931 \end{matrix}\right\}$$

Respectively. The slope of the yield curve is upward sloping; therefore, it can be concluded that the relationship between tenors and yields is directly proportional. In Table 2 below, the last quarter of 2021 is significant because there were additional tenors that appears as recorded. The quarter contains 13 maturities (tenors) which are One year, Two years, Four years, Six years, Seven years, Nine years, ten years, eleven years, twelve years, seventeen years, twenty six years, twenty eight years, and thirty years with corresponding yield as;

$$\left\{\begin{matrix} 2.1195,\ 3.6437,\ 5.2924,\ 6.1897,\ 6.4153,\ 7.1320,\ 7.6881,\ 7.7216, \\ 7.6664,\ 8.2340, 8.3439,\ 8.7784,\ \textit{and}\ 8.5303 \end{matrix}\right\}$$

respectively. The yield in this quarter as depicted by the figure above is consistently sloping upward with a decline on twelve years maturity. Therefore, we can conclude that as maturity increases, yield also increases.

Table 2: Fourth quarter descriptive statistics-2021

Tenors (τ)	N	Minimum	Maximum	Mean	Std. Deviation
One year	58	1.48	3.43	2.1195	.33856
Two years	58	2.94	4.36	3.6437	.34460
Four years	58	4.67	6.02	5.2924	.40861
Six years	58	5.76	6.85	6.1897	.31858
Seven years	58	5.98	7.14	6.4153	.33007
Nine years	58	6.75	7.85	7.1320	.31443
Ten years	58	7.26	8.39	7.6881	.33013
Eleven years	58	7.29	8.47	7.7216	.35613
Twelve years	58	7.19	8.45	7.6664	.34597
Seventeen years	58	7.80	8.88	8.2340	.31403
Twenty-six years	58	7.92	8.97	8.3439	.29639
Twenty-eight years	58	8.41	9.36	8.7784	.29491
Thirty years	58	8.12	9.15	8.5303	.30737

Source: Authors' computation

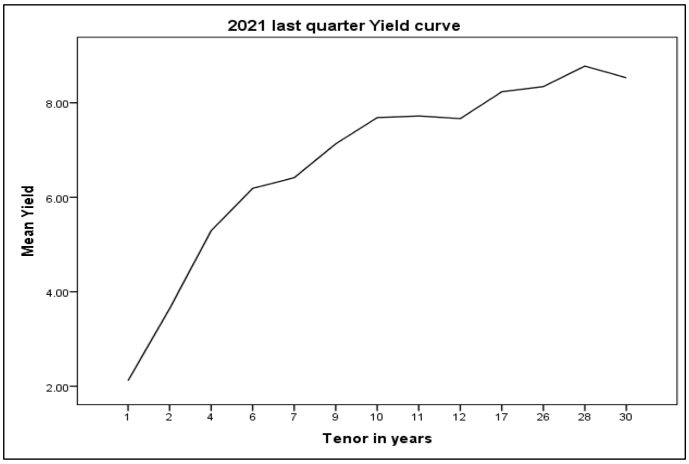

Figure 2: Fourth quarter yield curve

Figure 2 show the zero-coupon yield curve when the parameters are fitted to the average yield curve in the fourth quarter. The average yield to maturity is increasing over maturities, we then infer that the data set is consistent with the condition that the yield curve over time is increasing and concave. The essence is to justify and validate the construction of the extended Nelson-Siegel model.

Table 3: Descriptive statistics-2022

Tenors (τ)	N	Minimum	Maximum	Mean	Std. Deviation
One year	241	2.76	10.89	6.5488	2.02279
Three years	241	4.94	13.82	9.4788	2.60400
Five years	241	6.10	15.44	10.6853	2.57722
Six years	241	6.47	15.61	10.8068	2.52283
Eight years	241	7.28	15.99	11.2024	2.39675
Nine years	241	7.84	15.99	11.6304	2.24195
Ten years	241	8.01	16.88	11.6191	2.19466
Eleven years	241	.00	15.55	11.3677	2.21467
Sixteen years	241	8.50	15.21	11.7122	1.81228
Twenty-five years	241	8.60	14.83	11.4600	1.60907
Twenty-seven years	241	9.06	15.67	11.9866	1.70691
Twenty-nine years	241	.42	15.40	11.7561	1.88498

Source: Authors' computation

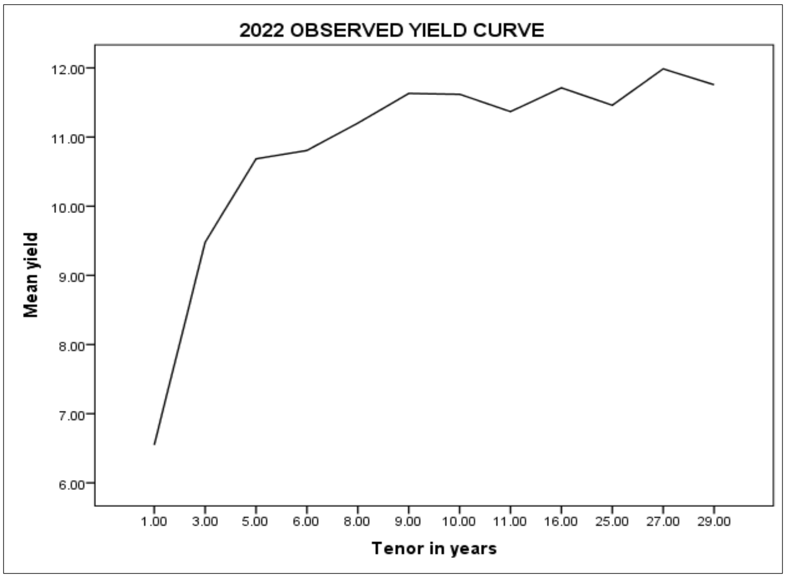

Figure 3: Observed yield curve for 2022

The 2022 statistics consisted of twelve tenors which are one year, three years, five years, six years, eight years, nine years, ten years, eleven years, sixteen years, twenty five years, twenty even years and twenty nine years which from the descriptive analysis has an average yield of

$$\begin{Bmatrix} 6.5488,\ 9.4788,\ 10.6853,\ 10.8068,\ 11.2024,\ 11.6304,\ 11.6191, \\ 11.3677,\ 11.7122, 11.4600,\ 11.9866\ \textit{and}\ 11.7561 \end{Bmatrix}$$

respectively. The overall statistics experience a higher level of changes which can be described as a measure of risk on the bond market. These can be attributed to the numerous data volatilities over the observed monthly market operations which provoked such behavior. It is imperative to note that the trading yield over the months are not the same. Some months have higher trading yield while others lower trading yield which significantly affect the variation in the data. The aggregate yield curve of observed yield for the year 2022 as depicted on the Figure 3 revealed that the yield curve is sloping upwards giving an increase of time to maturity. However, there was a decline as experienced on the nine years tenor to ten years before the observed upward movement and a decline in twenty five years tenor and a rise on twenty seven whereas there is a decline at the last tenor. Despite the noticeable fluctuations experienced over some tenors, we can conclude that the overall yield curve is upward sloping.

3.6. Predicting the In-Sample Yield of τ

The extended Nelson-Siegel four-factor model parameters as established in the previous section for yield analysis in respect of the observed data for the year 2021 and 2022 can be used for the prediction of in-sample tenors but not captured in the observed data given the parameters. The parameters were estimated using the ordinary least square method. For this study, we used the value 0.03778 and 0.0609 for λ_1 and λ_2 respectively. The results obtained are presented on table below.

Table 4: Model Parameters

First 3 quarters 2021		Unstandardized Coefficients		Standardized Coefficients	T	Sig.
		B	**Std. Error**	**Beta**		
1	β_1	8.388	.152		55.048	.000
	β_2	-6.104	.707	-.731	-8.635	.000
	β_3	3.451	2.150	.128	1.605	.160
	β_4	-8.911	3.227	-.379	-2.762	.033
Last quarter 2021		**Unstandardized Coefficients**		**Standardized Coefficients**	**T**	**Sig.**
		B	**Std. Error**	**Beta**		
1	β_1	9.312	.168		55.470	.000
	β_2	-9.166	.905	-1.035	-10.124	.000
	β_3	-.550	2.554	-.021	-.216	.834
	β_4	1.365	4.056	.057	.337	.744
2022		**Unstandardized Coefficients**		**Standardized Coefficients**	**T**	**Sig.**
		B	**Std. Error**	Beta		
1	β_1	11.879	.176		67.590	.000
	β_2	-8.165	.917	-1.164	-8.905	.000
	β_3	6.014	2.804	.318	2.145	.064
	β_4	.991	4.352	.054	.228	.826

Source: Authors' computation

From the Table 4, the parameters estimated were substituted into the model in order to estimate the in-sample maturities that were not captured on the observed data. Since the in-sample yield does not exceed a maturity of more than twenty-nine years and thirty years for 2021 and 2022 respectively, we write the model given a conditional statement for in-sample estimation of yield as follows: For the *first three quarters* 2021.

Let τ be the current time and suppose $f(u)$ represents the forward rate function at time $t + u$. Then the current price of a zero coupon bond maturing at par, at time $t + \tau$ is given by $\delta(\tau) = exp\left(- \int_0^\tau f(u)du\right)$. Consequently, the yield $y(\tau)$ and the price $p(\tau)$ of the bond as well as the forward rate function $f(u)$ all satisfy the relation $\tau y(\tau) = \ln p(\tau) = \int_0^\tau f(u)du$.

From the discount function $\delta_t(\tau) = exp\{-\tau y_t(\tau)\}$, we obtain

$$\delta_t(\tau) = \exp\left\{\begin{array}{l} -8.388\tau + 6.104\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau}\right)\tau \\ -3.451\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right)\tau \\ +8.911\left(\dfrac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right)\tau \end{array}\right\} \quad (55)$$

and the yield function is

$$y_t(\tau) = 8.388 - 6.104\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau}\right]$$
$$+3.451\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right]$$
$$-8.911\left[\frac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right] \quad (56)$$

where $0 \le \tau \le 336$. For the *fourth quarter* 2021

$$\delta_t(\tau) = \exp\left\{\begin{array}{l} -9.312\tau + 9.166\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau}\right)\tau \\ +0.55\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right)\tau \\ +1.361\left(\dfrac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right)\tau \end{array}\right\} \quad (57)$$

$$y_t(\tau) = 9.312 - 9.166\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau}\right] - 0.55\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right] -$$
$$1.361\left[\frac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right] \quad (58)$$

where $0 \le \tau \le 360$

For all quarters in 2022

$$\delta_t(\tau) = \exp\left[\left\{\begin{array}{l} -11.879\tau + 8.165\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau}\right)\tau \\ -6.014\left(\dfrac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right)\tau \\ -0.991\left(\dfrac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right)\tau \end{array}\right\}\right] \tag{59}$$

$$y_t(\tau) = 11.879 - 8.165\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau}\right] + 6.014\left[\frac{1-e^{-0.03778\tau}}{0.03778\tau} - e^{-0.03778\tau}\right] + 0.991\left[\frac{1-e^{-0.0669\tau}}{0.0669\tau} - e^{-0.0669\tau}\right] \tag{60}$$

$0 \leq \tau \leq 360$. The comparative yield for observed data and the in-sample prediction using the above model is presented below:

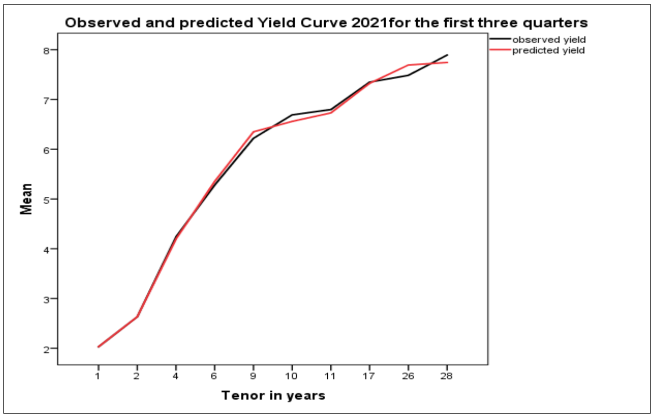

Figure 4: Predicted yield curves 2021 first three quarters

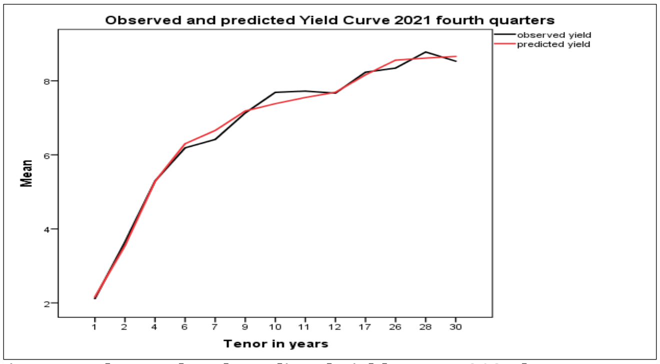

Figure 5: observed and predicted yield curves 2021 last quarter

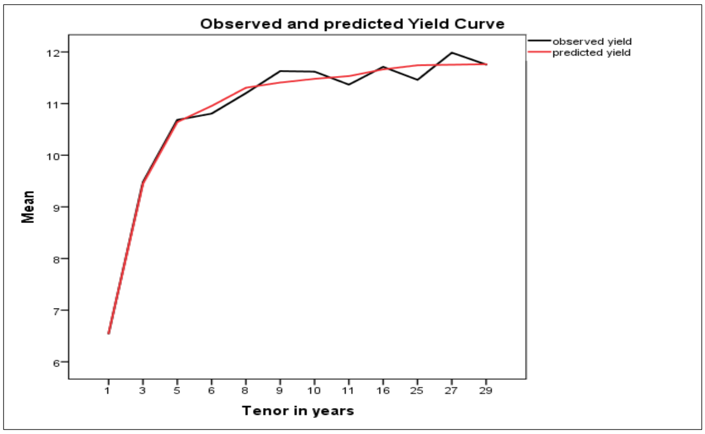

Figure 6: Observed and predicted yield curves 2022

3.7. Fitting the Model into the Observed Data

The measure of goodness of fit analysis is determined when ordinary least square method is applied on data by R square, R^2 adjusted and standard error estimation. The model measure of fit analysis is depicted in table below.

Table 5: Model Summary

First 3 quarters 2021	**R**	**R Square**	**Adjusted R Square**	**Std. Error of the Estimate**	**Durbin-Watson**
	.999[a]	.997	.996	.13653209	2.454
Last quarter 2021	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
	.997[a]	.994	.992	.18389683	1.982
2022	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
	.995[a]	.989	.985	.18325833	.989

Source: Authors' computation

The R square defines the degree in which the independent variable fits in the dependent variable, it determines how well the model was able to predict the observed dependent variable. For this study, the unobserved variable was able to define the observed yield with a high degree of accuracy. As seen from the table above, the R square for the first three quarters of 2021 , last quarter of 2021 and the overall yield of 2022 are 0.999, 0.997 and 0.995 respectively and the adjusted R square of the study scope as 0.996, 0.992 and 0.985. When using the results of adjusted R square values, it can be inferred that the model fits in the observed yield by 99% for the first three quarters of 2021, 99% for the last quarter 2021 and 98% for 2022. By these, the model can be used for estimation and prediction of yield.

3.8. The Effect of the Time to Maturity on Term Structure of Interest Rate

Theoretically, the relationship that existed between time to maturity or tenors of a set of a bond and its corresponding yield is referred to as the term structure of interest rate. To determine whether time to maturity has an effect on the term structure of interest rate, we used correlation between time to maturity and its corresponding yield as presented in the table 6 using the data set of 2021.

Table 6: Correlation Analysis

		Yield	**Tenor in Years**
Yield	Pearson Correlation	1	.847**
	Sig. (2-tailed)		.002
	N	10	10
Tenor in Years	Pearson Correlation	.847**	1
	Sig. (2-tailed)	.002	
	N	10	10

Source: Authors' computation

The Pearson correlation determines the degree to which the movement of tenors and yields are associated. The above table reveals that the correlation is significant given the confidence level of significance as 0.01. The correlation revealed a positive relationship between tenors and its corresponding yield. The correlation value of 0.847 means that 84.7% of the data explains the degree to which the tenor and yield correlate in a positive direction. The observed curves shown in this study also indicated that the slope of yield against its tenors is moving directly proportionally. That is, when the tenor increases, the corresponding yield also increases in percentage. We can conclude that time to maturity has a significant effect on the term structure of interest rate.

4. RESULTS AND DISCUSSION

From equation (37) using the first three quarters in 2021 for example, we have

$$PVFB = \sum_{i=1}^{T} \phi_i \left[\exp\left\{ \begin{array}{l} -8.388\tau_i + 6.104\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i}\right)\tau_i \\ -3.451\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i} - e^{-0.03778\tau_i}\right)\tau_i \\ +8.911\left(\dfrac{1-e^{-0.0669\tau_i}}{0.0669\tau_i} - e^{-0.0669\tau_i}\right)\tau_i \end{array} \right\} \right] \quad (61)$$

While for the fourth quarter in 2021.

$$PVFB = \sum_{i=1}^{T} \phi_i \left[\exp \left\{ \begin{array}{l} -9.312\tau_i + 9.166\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i}\right)\tau_i \\ +0.55\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i} - e^{-0.03778\tau}\right)\tau_i \\ +1.361\left(\dfrac{1-e^{-0.0669\tau_i}}{0.0669\tau_i} - e^{-0.0669\tau}\right)\tau_i \end{array} \right\} \right] \quad (62)$$

However, using the combined 4 quarters in 2022, we have

$$PVFB = \sum_{i=1}^{T} \phi_i \left[\exp \left[\left\{ \begin{array}{l} -11.879\tau_i + 8.165\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i}\right)\tau_i \\ -6.014\left(\dfrac{1-e^{-0.03778\tau_i}}{0.03778\tau_i} - e^{-0.03778\tau_i}\right)\tau_i \\ -0.991\left(\dfrac{1-e^{-0.0669\tau_i}}{0.0669\tau_i} - e^{-0.0669\tau_i}\right)\tau_i \end{array} \right\} \right] \right] \quad (63)$$

The analysis of the Nigerian Eurobond yield for the years 2021 and 2022 includes descriptive analysis for the daily trading of the Nigerian Eurobond, the model analysis of extended Nelson-Siegel four factor model as well as correlation analysis. From the descriptive statistics obtained, the mean yield increases in direct proportional to tenor and consequently the yield increases as tenor increases. By this, we can confirm the market condition that the higher the risk, the higher the expected return. The risk in this case involves the long-term maturity that there is a higher risk on long-term securities in general. These risks include interest rate changes and rising inflation that depreciates the value of money value. The extended Nelson-Siegel four factor model was estimated using the quadratic loss function method. The unobserved variables become the independent variable whereas the observed yield becomes the dependent variable. The results of the four-factor parameters computed were substituted into the model so as to obtain a solution for predicting the in-sample yield of the tenors not captured in the observed yield. The model goodness of fit was tested in order to find how well the model fits in the observed data yield to assume its prediction capability. The output result came out with a high level of fit which was evidence from the R square adjusted for both 2021 and 2022 given an average of 99%. From this we can conclude that the extended Nelson-Siegel four factor model can be used for the analysis and prediction of yield. The correlation analysis was able to verify the relationship that existed between tenors and their corresponding yield. The result shows a positive relationship which confirms the direct proportionality. Figures 7-14 describes the trajectories of the predicted model parameters while Tables 7-14 describe the values of the predicted model parameters.

4.1. The Impact of the Console and Short Rate on the Associated Exponential Terms

The following limiting results were obtained from the coefficients of β_i.

$$\lim_{\tau\to 0}\frac{\lambda_1}{\tau}\left[1-e^{-\frac{\tau}{\lambda_1}}\right]=\lim_{\tau\to 0}\frac{\lambda_1}{\tau}\times\lim_{\tau\to 0}\left(1-e^{-\frac{\tau}{\lambda_1}}\right)=\lim_{\tau\to 0}\frac{0}{1}\times\lim_{\tau\to 0}\left(\frac{1}{\lambda_1}e^{-\frac{\tau}{\lambda_1}}\right)=0 \quad (63a)$$

If $Z=e^{-\frac{\tau}{\lambda_t}}=e^{-\tau\times\lambda_t^{-1}}$ (64)

$$\frac{dZ}{d\lambda_t}=-\tau\lambda_t^{-2}e^{-\tau\times\lambda_t^{-1}} \quad (65)$$

$$\lim_{\lambda_t\to 0}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}\right]=\lim_{\lambda_t\to 0}\left[-\frac{\tau\lambda_t^{-2}e^{-\tau\times\lambda_t^{-1}}}{-\tau\lambda_t^{-2}}\right] \quad (66)$$

$$\lim_{\lambda_t\to 0}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}\right]=\lim_{\lambda_t\to 0}e^{-\frac{\tau}{\lambda_t}}=0 \quad (67)$$

$$\lim_{\lambda_t\to \infty}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}\right]=e^{0}=1 \quad (68)$$

$$\lim_{\lambda_t\to 0}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_t}}\right]=\lim_{\lambda_t\to 0}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}\right]-\lim_{\lambda_t\to 0}e^{-\frac{\tau}{\lambda_t}} \quad (69)$$

$$\lim_{\lambda_t\to 0}\left[\frac{1-e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}}-e^{-\frac{\tau}{\lambda_t}}\right]=\lim_{\lambda_t\to 0}e^{-\frac{\tau}{\lambda_t}}-\lim_{\lambda_t\to 0}e^{-\frac{\tau}{\lambda_t}} \quad (70)$$

$$\lim_{\lambda_t \to 0} e^{-\frac{\tau}{\lambda_t}} = 0 \tag{71}$$

$$\lim_{\lambda_t \to \infty} \left[\frac{1 - e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}} - e^{-\frac{\tau}{\lambda_t}} \right] = \lim_{\lambda_t \to \infty} \left[\frac{1 - e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}} \right] - \lim_{\lambda_t \to \infty} e^{-\frac{\tau}{\lambda_t}} \tag{72}$$

$$\lim_{\lambda_t \to \infty} \left[\frac{1 - e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}} - e^{-\frac{\tau}{\lambda_t}} \right] = \lim_{\lambda_t \to \infty} e^{-\frac{\tau}{\lambda_t}} - \lim_{\lambda_t \to \infty} e^{-\frac{\tau}{\lambda_t}} \tag{73}$$

$$\lim_{\lambda_t \to \infty} \left[\frac{1 - e^{-\frac{\tau}{\lambda_t}}}{\frac{\tau}{\lambda_t}} - e^{-\frac{\tau}{\lambda_t}} \right] = 0 \tag{74}$$

4.2. Analysis Using Different Values of the Model Parameters

4.2.1. Predicted Yield at Different Values of β_1

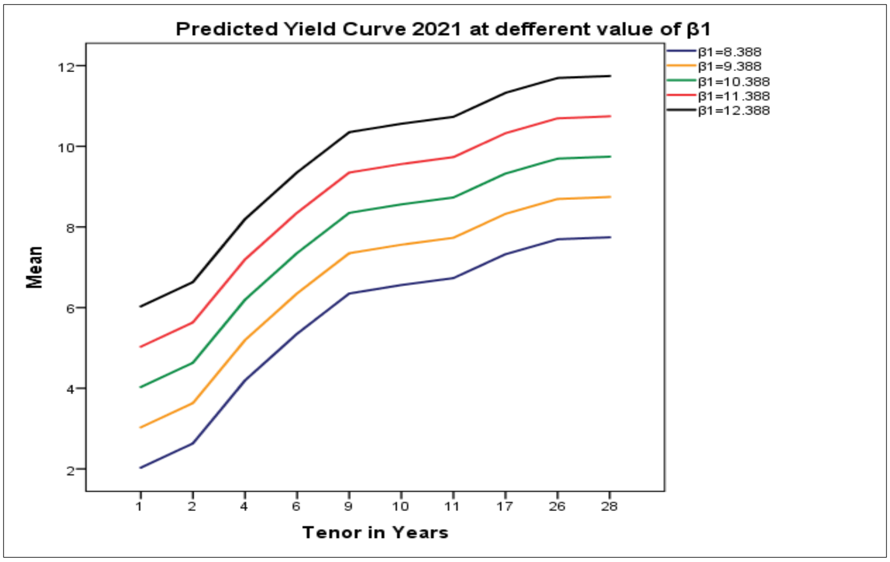

Figure 7: Predicted yield curve for 2021 at different values of β_1

Table 7: Different values of β_1

β_1=8.388	β_1=9.388	β_1=10.388	β_1=11.388	β_1=12.388
2.030682	3.030682	4.030682	5.030682	6.030682
2.633049	3.633049	4.633049	5.633049	6.633049
4.195382	5.195382	6.195382	7.195382	8.195382
5.353794	6.353794	7.353794	8.353794	9.353794
6.349906	7.349906	8.349906	9.349906	10.34991
6.55947	7.55947	8.55947	9.55947	10.55947
6.730821	7.730821	8.730821	9.730821	10.73082
7.325151	8.325151	9.325151	10.32515	11.32515
7.693924	8.693924	9.693924	10.69392	11.69392
7.743513	8.743513	9.743513	10.74351	11.74351

Source: Authors' computation

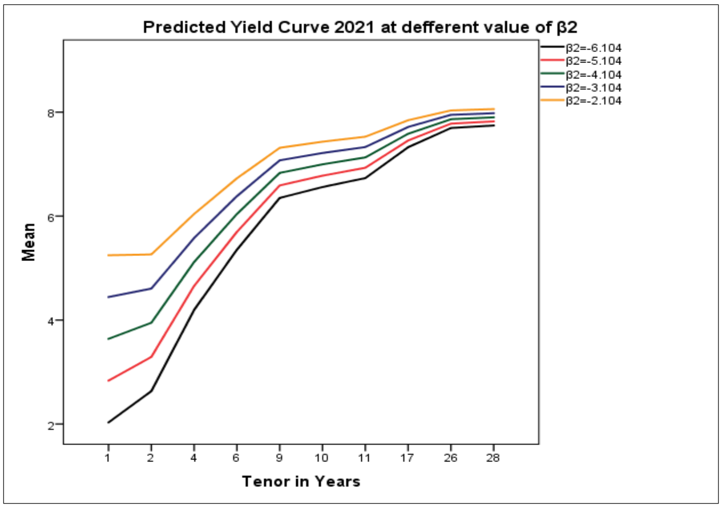

Figure 8: Predicted yield curve for 2021 at different values of β_2

Table 8: Different values of β_2

β_2=-6.104	β_2=5.104	β_2=-4.104	β_2=-3.104	β_2=-2.104
2.030682	2.834703	3.638723	4.442744	5.246764
2.633049	3.290532	3.948016	4.605499	5.262982
4.195382	4.656885	5.118388	5.579891	6.041394
5.353794	5.697206	6.040618	6.384031	6.727443
6.349906	6.590847	6.831787	7.072728	7.313669
6.55947	6.777676	6.995881	7.214087	7.432293
6.730821	6.929975	7.129129	7.328283	7.527437
7.325151	7.454843	7.584534	7.714226	7.843918
7.693924	7.77876	7.863596	7.948432	8.033268
7.743513	7.82229	7.901067	7.979843	8.05862

Source: Authors' computation

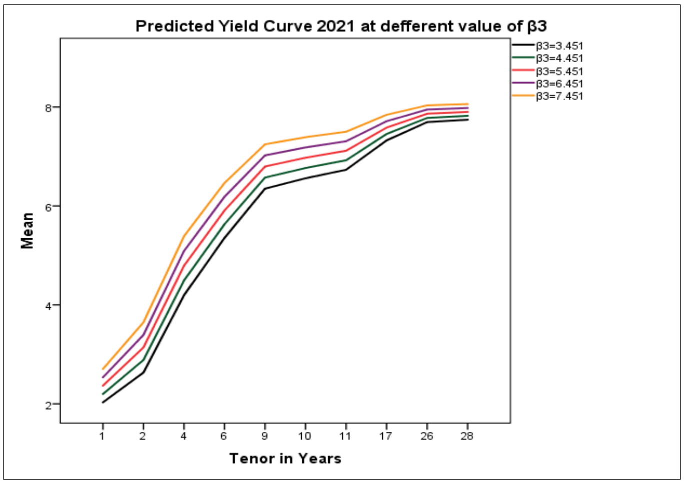

Figure 9:Predicted yield curve for 2021 at different values of β_3

Table 9: Different values of β_3

β_3=3.451	β_3=4.451	β_3=5.451	β_3=6.451	β_3=7.451
2.030682	2.199213	2.367745	2.536276	2.704807
2.633049	2.886685	3.140322	3.393959	3.647596
4.195382	4.493793	4.792204	5.090615	5.389026
5.353794	5.631342	5.90889	6.186438	6.463986
6.349906	6.573943	6.797981	7.022018	7.246056
6.55947	6.766934	6.974398	7.181861	7.389325
6.730821	6.923148	7.115476	7.307804	7.500132
7.325151	7.454393	7.583635	7.712877	7.84212
7.693924	7.778752	7.86358	7.948409	8.033237
7.743513	7.822287	7.90106	7.979834	8.058608

Source: Authors' computation

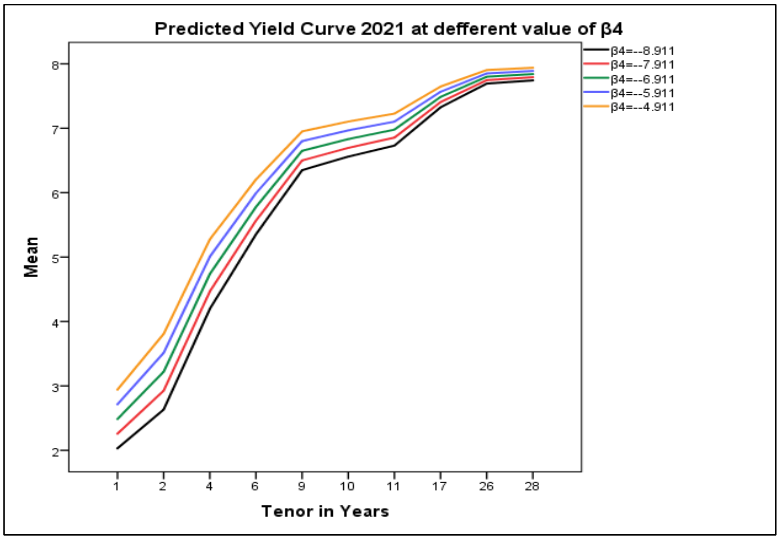

Figure 10: Predicted yield curve for 2021 at different values of β_4

Table 10: Different values of β_4

β_4=-8.911	β_4=-8.911	β_4=-8.911	β_4=-8.911	β_4=-8.911
2.030682	2.258623	2.486563	2.714504	2.942444
2.633049	2.926728	3.220407	3.514086	3.807764
4.195382	4.465321	4.735259	5.005197	5.275135
5.353794	5.566547	5.779299	5.992051	6.204804
6.349906	6.500343	6.65078	6.801217	6.951654
6.55947	6.695544	6.831619	6.967693	7.103768
6.730821	6.854855	6.978888	7.102922	7.226956
7.325151	7.405638	7.486126	7.566614	7.647101
7.693924	7.746553	7.799182	7.851812	7.904441
7.743513	7.792383	7.841254	7.890124	7.938994

Source: Authors' computation

4.2.2. Predicted Yield at Different Values of β_1

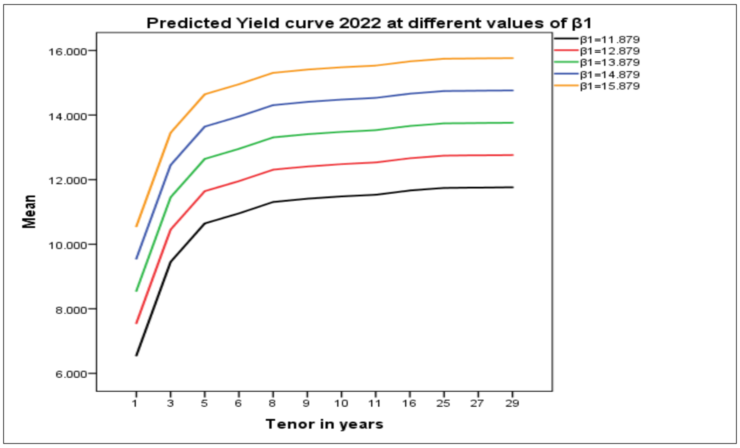

Figure 11: Predicted yield curve for 2022 at different values of β_1

Table 11: Different values of β_1

$\beta_1$1=1.879	$\beta_1$1=2.879	$\beta_1$1=3.879	$\beta_1$1=4.879	$\beta_1$1=5.879
6.553608	7.553608	8.553608	9.553608	10.55361
9.450829	10.45083	11.45083	12.45083	13.45083
10.64366	11.64366	12.64366	13.64366	14.64366
10.95505	11.95505	12.95505	13.95505	14.95505
11.30789	12.30789	13.30789	14.30789	15.30789
11.40816	12.40816	13.40816	14.40816	15.40816
11.47989	12.47989	13.47989	14.47989	15.47989
11.53248	12.53248	13.53248	14.53248	15.53248
11.66316	12.66316	13.66316	14.66316	15.66316
11.74339	12.74339	13.74339	14.74339	15.74339
11.75347	12.75347	13.75347	14.75347	15.75347
11.76214	12.76214	13.76214	14.76214	15.76214

Source: Authors' computation

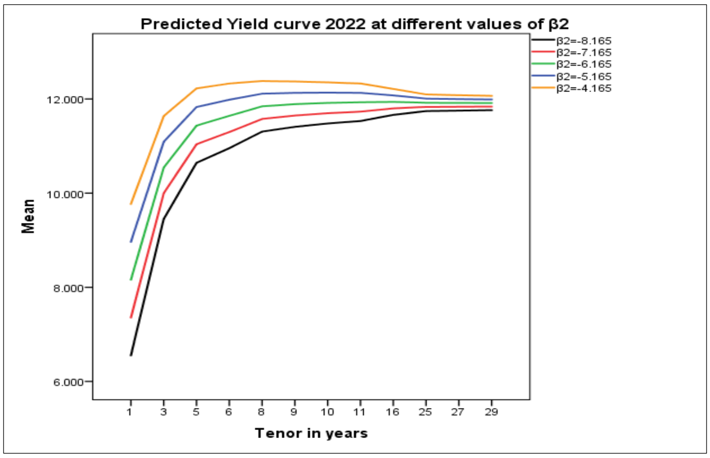

Figure 12: Predicted yield curve for 2022 at different values of β_2

Table 12: Different values of β_2

β_2-=8.165	β_2-=7.165	β_2=-6.165	β_2-5.165	β_2-=4.165
6.553608	7.357629	8.161649	8.96567	9.76969
9.450829	9.997385	10.54394	11.0905	11.63705
10.64366	11.03909	11.43451	11.82994	12.22537
10.95505	11.29846	11.64187	11.98529	12.3287
11.30789	11.57627	11.84466	12.11304	12.38143
11.40816	11.6491	11.89004	12.13098	12.37193
11.47989	11.69809	11.9163	12.1345	12.35271
11.53248	11.73164	11.93079	12.12995	12.3291
11.66316	11.80093	11.93869	12.07645	12.21421
11.74339	11.83162	11.91985	12.00808	12.09631
11.75347	11.83516	11.91686	11.99855	12.08025
11.76214	11.8382	11.91426	11.99032	12.06638

Source: Authors' computation

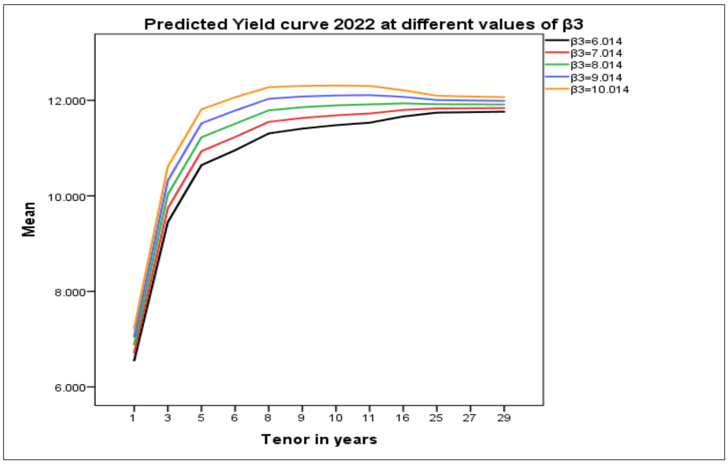

Figure 13: Predicted yield curve for 2022 at different values of β_3

Table 13: Different values of β_3

β_3=6.014	β_3=7.014	β_3=8.014	β_3=9.014	β_3=10.014
6.553608	6.72214	6.890671	7.059202	7.227733
9.450829	9.740745	10.03066	10.32058	10.61049
10.64366	10.93544	11.22723	11.51901	11.8108
10.95505	11.2326	11.51015	11.78769	12.06524
11.30789	11.54967	11.79146	12.03325	12.27503
11.40816	11.6322	11.85624	12.08027	12.30431
11.47989	11.68735	11.89481	12.10228	12.30974
11.53248	11.72481	11.91714	12.10947	12.30179
11.66316	11.80022	11.93727	12.07433	12.21138
11.74339	11.83161	11.91982	12.00804	12.09626
11.75347	11.83516	11.91685	11.99854	12.08023
11.76214	11.8382	11.91426	11.99032	12.06638

Source: Authors' computation

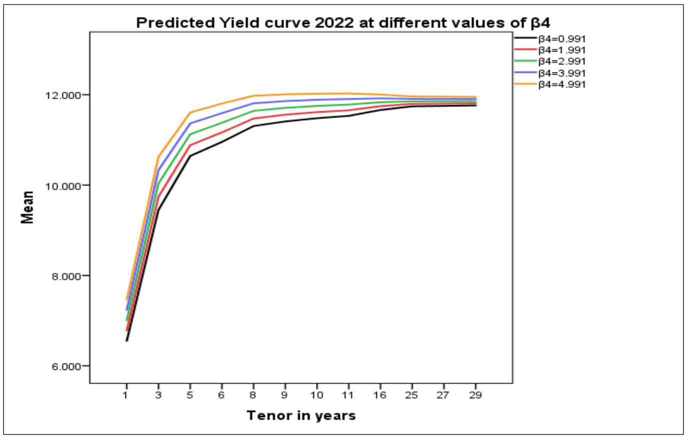

Figure 14: Predicted yield curve for 2022 at different values of β_4

Table 14: Different values of β_4

β_4 =0.991	β_4 =1.991	β_4 =2.991	β_4 =3.991	β_4 =4.991
6.553608	6.781549	7.009489	7.23743	7.46537
9.450829	9.744377	10.03792	10.33147	10.62502
10.64366	10.88436	11.12506	11.36576	11.60646
10.95505	11.1678	11.38056	11.59331	11.80606
11.30789	11.47555	11.64321	11.81087	11.97853
11.40816	11.5586	11.70904	11.85947	12.00991
11.47989	11.61596	11.75204	11.88811	12.02418
11.53248	11.65652	11.78055	11.90458	12.02862
11.66316	11.74868	11.83419	11.9197	12.00522
11.74339	11.79812	11.85286	11.90759	11.96233
11.75347	11.80415	11.85483	11.90551	11.95619
11.76214	11.80933	11.85651	11.9037	11.95088

Source: Authors' computation

5. CONCLUSION

The application of forward interest rates serves to be a benchmark financial tool in pricing new market instruments. Though the adoption of yield curves is recognized in monetary policies, its use in financial analysis is yet to be discovered in Nigeria market. Motivated by the central role assumed by the term structure of interest rates in the Nigerian economy, we examined the forward rate function of the extended Nelson-Siegel parsimonious function through comprehensive mathematical analysis when applied to the Nigerian Eurobond. The forward rate function then becomes the instantaneous interest rate locked at a previous time ξ for a subsequent investment decision. In this paper, the analysis was conducted under the following key result areas (i) the discount function and applied on the present value of benefits (ii) the yield function. From the parsimonious perspectives, we explained the use of time varying exponentially decay terms to ascertain more flexible parsimonious parameters so as to obtain accurate estimating results and to reflect the market operators' views on interest rate levels in a forward looking dimension. Obviously, the efficacy of the extended Nelson-Siegel model is a function of how precise the model fits the yield curve at each date and on how well the factors can be forecast from their resulting time series. Consequently, we highlight the following weaknesses. The extension is challenging as there are two λ's that were estimated and the analytical form suffers from the problem of degeneracy where the factors λ_1 and λ_2 are equivalent. Furthermore, when $\lambda_1 \neq \lambda_2$, the specified functional form results in an underlying parameter estimation difficulty which is no longer mathematically convenient because it will hence involve many dimensional minimization. Despite the weakness observed, the extended Nelson-Siegel model is notably distinguished through its empirical analysis of the parameters which change with time. The straight forward analysis of its few parameters estimation seems consistent with the illiquid and under developed markets such as Nigeria. Relatively, in this study, it has been used for the construction of a smooth surface curve and for the modelling of different deformations of the curve. From the computed results of the highly flexible estimation technique, we emphasized the in-sample potential of the extended Nelson-Siegel in the Euro-bond market and recommend that the parsimonious model can be presented as an interventionist investment tool during volatile market conditions.

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